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Edge waves over a shelf: full linear theory

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Edge-wave solutions to the linearized shallow-water equations for water waves are well known for a variety of bottom topographies. The only explicit solution using the full linearized theory describes edge waves over a uniformly sloping beach, although the existence of such waves has been established for a wide class of bottom geometries. In this paper the full linearized theory is used to derive the properties of edge waves over a shelf. In particular, curves are presented showing the variation of frequency with wavenumber along the shelf, together with some mode shapes for a particular shelf geometry.

1. Introduction

The best-known and oldest example of a wave that can propagate unchanged along a straight coastline is that in water of constant bottom slope $\tan\alpha$ discovered by Stokes (1846). The longshore wavenumber l is related to the wave frequency σ by $\sigma^2 g^{-1} (\equiv K) = l \sin\alpha$. The amplitude of the wave decreases exponentially with distance out to sea, so that the energy is effectively trapped near the shoreline. Ursell (1952) provided a class of such edge waves of which the Stokes wave is just the first and whose frequencies are given by $K = l \sin(2n+1)\alpha$ ($n = 0, 1, \ldots$). Roseau (1958), apparently unaware of Ursell's work, developed a systematic technique for constructing edge-wave solutions over a uniformly sloping bottom, using Laplace-type integrals and functional equations. In addition to the bounded edge waves of Ursell, he constructed edge waves having frequencies satisfying $K = l \sin 2n\alpha$ ($n = 1, 2, \ldots$), which are characterized by potentials having logarithmic singularities at the intersection of the mean free surface and the bottom. Edge-wave solutions possessing higher-order singularities were also constructed by Roseau.

The Ursell edge waves remain the only explicit bounded solutions based on the full linearized theory, although the *existence* of such waves in a wide class of problems was shown by Jones (1953). For example, he showed there exists at least one edge wave on the horizontal part of a submerged rectangular protuberance projecting outwards from the side of a vertical wall, in either finite or infinite depth of water. Furthermore, the proof is independent of the depth, size, or shape of the protuberance, provided that it is submerged. In particular, using symmetry arguments, it follows that there exist waves that are trapped over the top of a submerged cylinder of any radius in deep water and indeed over the top of any symmetric ridge on the sea bed. The case of trapped waves over the top of a submerged circular cylinder was studied separately by Ursell (1951), who showed that such trapped waves exist, at least for sufficiently small cylinders.

Grimshaw (1974) considered the lowest edge-wave mode, where both l and $K \to 0$ together, and obtained upper and lower bounds for the dispersion relation between K and l for a class of bottom profiles in which the water depth approaches a constant value at large distances from the coast.

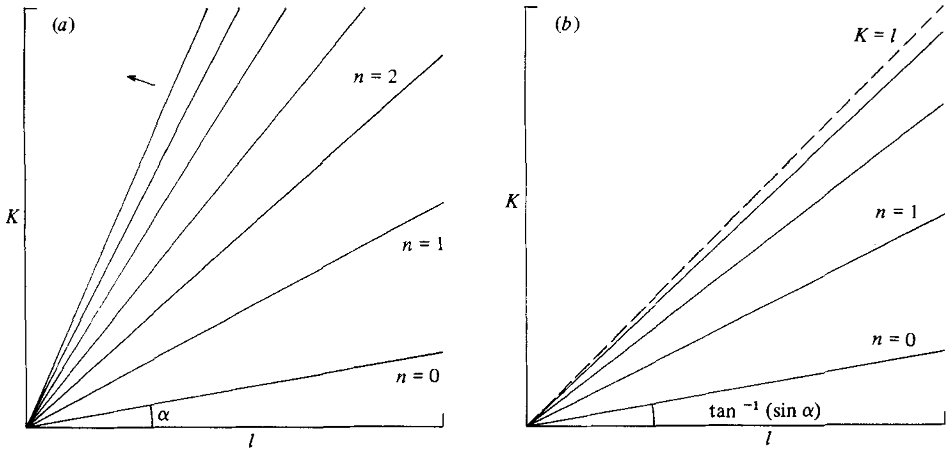

FIGURE 1. Dispersion relations for edge waves on a plane beach of slope $\tan\alpha$: (a) shallow-water theory due to Eckert (1951), (b) full linear theory due to Ursell (1952).

If the shallow-water approximation is employed, whereby $Kh \ll 1$, where h is the depth of the bottom, explicit solutions are possible for a variety of bottom topographies. For example, for a uniformly sloping bottom of fixed slope α, Eckart (1951) showed that, for either l or K fixed, there exists an *infinite* set of edge wave modes satisfying

$$K = (2n+1)\, l \tan\alpha \quad (n = 0, 1, 2, \ldots), \tag{1.1}$$

a result that agrees with Ursell's edge waves as $\alpha \to 0$. However, Ursell's full linear theory predicts a *finite* number of modes given by the greatest integer contained in $\frac{1}{2} + \pi/4\alpha$. The situation is illustrated in figure 1.

The problem considered here is that of edge waves travelling along a submerged horizontal shelf bounded on one side by a vertical wall extending through the free surface, and on the other by a vertical drop from the shelf to a deeper region of constant water depth extending horizontally indefinitely. The shallow-water dispersion relation for edge-wave modes for this problem has been extensively studied by Snodgrass, Munk & Miller (1962), Summerfield (1972) and Longuet-Higgins (1967), and it can be shown that, for a *fixed* geometry, the number of modes increases indefinitely with increasing K. Here the problem is considered using both approximate theories and also full linear theory. One question which immediately arises is whether the number of edge-wave modes is finite or infinite for a given shelf geometry when a more accurate full linear theory is used.

The problem is considered in stages. In §2 the usual linearized equations and boundary conditions to be satisfied by the velocity potential are presented, on the assumption of an inviscid incompressible irrotational fluid. The simple shallow-water theory is reproduced in §3, showing how the number of modes increases with increasing frequency. In §4 a shallow–deep approximation is employed in which the shallow-water solution valid over the shelf is matched to a deep-water solution extending out to sea where the linear *infinite*-depth solution applies. The matching takes place via an intermediate region in the vicinity of the edge of the shelf. The technique is identical with that used by Tuck (1980) and Newman, Sortland & Vinje (1983) for solving similar problems. A dispersion relation is derived connecting l and K, from which the edge-wave modes can be computed easily. It turns out that in contrast with the shallow-water solutions the number of edge-wave modes for a given geometry does *not* increase indefinitely with frequency, but has a maximum with just a single edge-wave mode at both large and small frequencies.

Neither of these approximate solutions is valid for $Kh \gg 1$, and in §5 the full linear equations are considered, and by matching appropriate eigenfunctions across the edge of the shelf a homogeneous infinite system of equations is obtained for the coefficients in the edge-wave eigenfunction expansion. Edge-wave modes correspond to the vanishing of the determinant of the system. Computation of the zeros of this infinite determinant is facilitated by the known approximate solutions derived in §§3 and 4 and also by a simple approximate variational approach to an equivalent integral equation.

Confirmation of the number of edge-wave modes is provided by the work of Jones (1953), from which upper and lower bounds on the number of modes to be expected for particular values of the parameters can be deduced. It is found that, on the full linear theory also, the number of edge-wave solutions is bounded for a given geometry as K varies, in contrast with the shallow-water approximation.

Curves are presented showing the variation of longshore wavenumber with wave frequency for the edge-wave modes on both the approximate and full linear theories. Typical surface elevations of these latter modes are also presented.

2. Formulation

Cartesian axes are chosen with the mean free surface the (x, z)-plane, z being directed along the straight coastline and y vertically downwards as shown in figure 2. The shallower water is of depth h_1 above the horizontal shelf of width a; the deeper water is of depth h_2.

On the basis of the usual assumptions of an inviscid incompressible fluid there exists a velocity potential $\Phi(x, y, z, t)$, which, assuming the linear theory of irrotational surface waves, satisfies

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{in } S\text{, the water region,} \tag{2.1}$$

with boundary conditions

$$\frac{\partial^2 \Phi}{\partial t^2} = g\frac{\partial \Phi}{\partial y} \quad \text{on} \quad y = 0, \quad x \geqslant 0, \tag{2.2}$$

$$\frac{\partial \Phi}{\partial y} = 0 \quad \begin{cases} \text{on} \quad y = h_1, \quad 0 \leqslant x < a, & (2.3a) \\ \text{on} \quad y = h_2, \quad x > a, & (2.3b) \end{cases}$$

$$\frac{\partial \Phi}{\partial x} = 0 \quad \begin{cases} \text{on} \quad x = 0, \quad 0 \leqslant y < h_1, & (2.4a) \\ \text{on} \quad x = a, \quad h_1 < y \leqslant h_2. & (2.4b) \end{cases}$$

Since we are concerned with progressive waves that are periodic in the longshore direction, we write

$$\Phi(x, y, z, t) = \phi(x, y)\cos(lz - \sigma t), \tag{2.5}$$

so that ϕ satisfies

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = l^2\phi \quad \text{in } S, \tag{2.6}$$

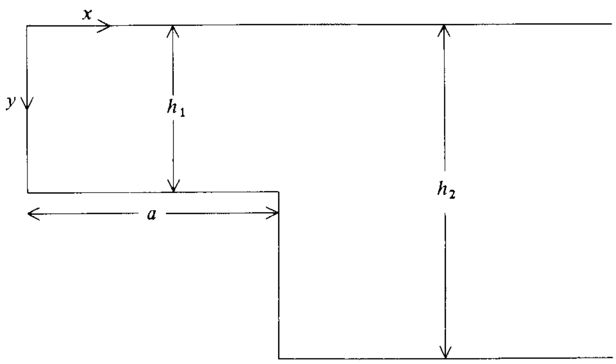

FIGURE 2. Definition sketch.

with

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on} \quad y = 0, \quad x \geqslant 0, \tag{2.7}$$

where $K = \sigma^2/g$.

Conditions (2.3) and (2.4) are also satisfied by $\phi(x, y)$.

For edge-wave solutions, we assume

$$\phi, \nabla\phi \to 0 \quad \text{as} \quad x \to \infty. \tag{2.8}$$

The aim is to find a dispersion relation between the wave frequency σ and the long-shore wavenumber l, such that non-trivial solutions to the above equations exist.

3. The shallow-water approximation

A number of authors have solved the problem under the assumption $Kh_i \ll 1$ $(i = 1, 2)$, so that the wavelength is large compared with each depth. With this assumption the solution for $\phi(x, y)$ is a potential $\psi(x)$ $(\equiv \phi(x, 0))$ given by

$$\psi(x) = \psi_1(x) = \frac{\cos px}{\cos pa} \quad (0 < y < h_1, \; 0 < x < a), \tag{3.1}$$

$$\psi(x) = \psi_2(x) = e^{-q(x-a)} \quad (0 < y < h_2, \; x > a), \tag{3.2}$$

where

$$p = (k_1^2 - l^2)^{\frac{1}{2}}, \quad q = (l^2 - k_2^2)^{\frac{1}{2}} \tag{3.3}$$

and

$$K = k_i^2 h_i \quad (i = 1, 2). \tag{3.4}$$

In addition, p and q are related by

$$p \tan pa = \mu^2 q, \quad \text{where} \quad \mu^2 = h_2/h_1. \tag{3.5}$$

Full details of the solution procedure may be found in Snodgrass *et al.* (1962).

The required conditions on ψ_1 are also satisfied if $l > k_1$, but in this case the left-hand side of (3.5) becomes $-p \tanh pa$, which is negative for $p \neq 0$, so that no edge-wave solution of (3.5) with $q > 0$ exists. It follows from (3.3) and (3.4) that

$$k_2 < l < k_1, \quad h_2 > h_1 \tag{3.6}$$

are necessary conditions for edge-wave solutions on the basis of shallow-water theory.

In figure 3 we sketch pa against qa as given by the relation (3.5) for a particular

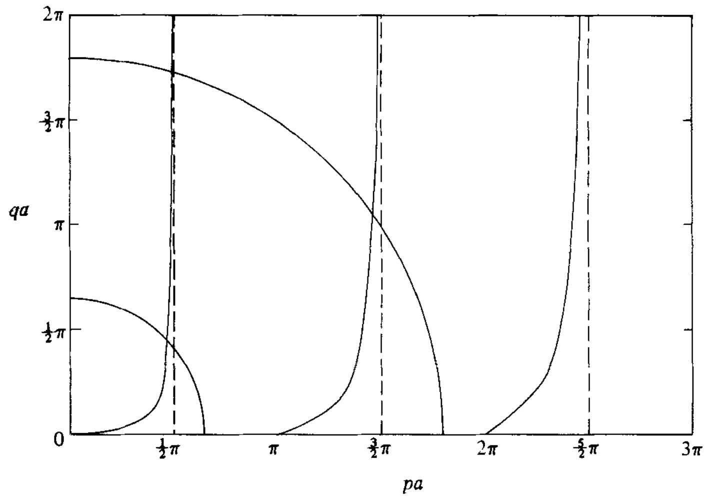

FIGURE 3. Sketch of the relationship between pa and qa as given by (3.5) for $\mu^2 = 10$. The circular arcs are typical lines of constant frequency (see (3.8)).

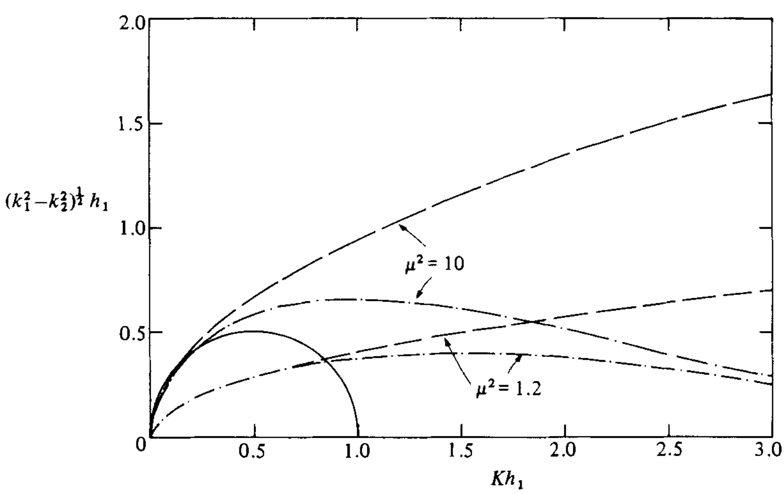

FIGURE 4. Variation of $(k_1^2-k_2^2)^{\frac{1}{2}} h$ with the frequency parameter Kh_1 for $\mu^2 = 1.2$, 10: ----, shallow-water theory; ——, shallow/deep approximation; -·-·-·, full linear theory.

μ^2. Notice that all edge-wave solutions have values of the offshore wavenumber pa in the ranges

$$(n-1)\pi < pa < (n-\tfrac{1}{2})\pi \quad (n = 1, 2, \ldots). \tag{3.7}$$

Now, from (3.3)

$$(p^2+q^2)\,a^2 = (k_1^2-k_2^2)\,a^2, \tag{3.8}$$

so that for a fixed frequency, possible solutions lie along circles in the (p, q)-plane (see figure 3). It is evident that there are just n edge-wave solutions when

$$(n-1)\pi \leqslant (k_1^2-k_2^2)^{\frac{1}{2}} a \leqslant n\pi. \tag{3.9}$$

From (3.4)

$$(k_1^2-k_2^2)\,h_1^2 = Kh_1(1-\mu^{-2}), \tag{3.10}$$

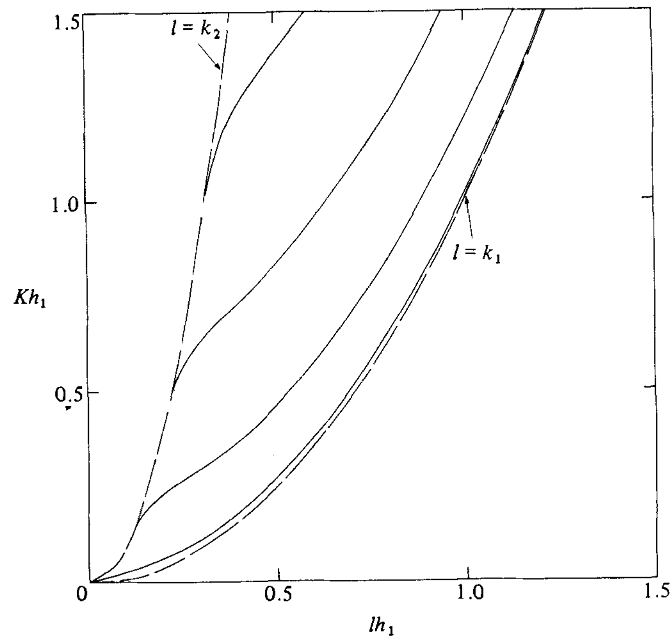

FIGURE 5. The shallow-water dispersion relation for $\mu^2 = 10$, $a/h_1 = 10$.

and figure 4 shows the variation of $(k_1^2 - k_2^2)^{\frac{1}{2}} h_1$ with Kh_1 for two values of μ^2. The precise number of modes for a given Kh_1 follows from figure 4 and (3.9). To determine the dispersion relation between lh_1 and Kh_1, p and q are eliminated from (3.5) using (3.3) and (3.4). The result is shown in figure 5. It can be seen from figure 5 that, although the number of edge-wave modes increases indefinitely with Kh_1, so does the longshore wavenumber lh_1, whilst for a *fixed* lh_1 there are only a *finite* number of modes, in contrast with Eckart's shallow-water dispersion relation (1.1) for a uniformly sloping beach. Further details of the shallow-water theory, together with diagrams showing the surface profiles of the edge-wave modes, are to be found in Snodgrass *et al.* (1962). A careful consideration of the validity of linear shallow-water theory, especially in the vicinity of the shelf, is given by Tuck (1976).

The validity of the results for moderate or large Kh must be regarded as doubtful, as the basic assumption of shallow-water theory is no longer satisfied.

4. The shallow/deep approximation

In this section the assumption of shallow-water theory in the region $x > a$ is relaxed, and the full linear theory used there. In fact, for simplicity it will be assumed that $Kh_1 \ll 1$, $Kh_2 \gg 1$, with similar assumptions for l. Thus the shallow-water solution in the region over the shelf is matched with a full linear solution for infinitely deep water, through an intermediate solution valid in a region close to the edge of the shelf. A similar matching technique has been employed by Tuck (1980) and Newman *et al.* (1983) for related problems. The method of the latter is followed closely here.

Two outer regions are considered where wave effects are important and one inner region where they are not. In the internal outer region $0 < y < h_1$, $0 < x \ll a$ the shallow-water solution

$$\psi_1 = \frac{\cos px}{\cos pa} \tag{3.1}$$

is valid.

The inner region is $x-a = O(h_1), y = O(h_1)$, at the vicinity of the edge of the shelf. The use of appropriate inner coordinates $X = (x-a)/h_1$, $Y = y/h_1$ then shows that the potential in the inner region satisfies

$$\Psi_{XX} + \Psi_{YY} = 0 \quad \text{in the fluid,} \tag{4.1}$$

$$\Psi_Y = 0 \quad \begin{cases} (Y = 0, \ -\infty < X < \infty), & (4.2) \\ (Y = 1, \ X < 0), & (4.3) \end{cases}$$

$$\Psi_X = 0 \quad (X = 0, \ Y > 1), \tag{4.4}$$

where terms involving Kh_1, lh_1 have been neglected. Thus in the inner region the problem reduces to the strictly two-dimensional flow from $X = -\infty$ between two parallel rigid walls out into the quarter-plane $X > 0$, $Y > 1$. The solution is given by Newman *et al.* (1983) using the conformal mapping

$$Z(= X + \mathrm{i}Y) = \frac{2}{\pi}(1-w)^{\frac{1}{2}} + \frac{1}{\pi}\ln\frac{(1-w)^{\frac{1}{2}}-1}{(1-w)^{\frac{1}{2}}+1}$$

to transform the fluid region in the complex z-plane into the lower half of the complex w-plane. In particular, they show that

$$\Psi \sim \frac{2m}{\pi}\ln\left(\frac{\pi R}{2}\right) + C \quad \text{as} \quad R = (X^2 + Y^2)^{\frac{1}{2}} \to \infty, \quad \text{for} \quad X > 0, \quad Y > 1, \tag{4.5a}$$

whilst

$$\Psi \sim m\{X - 2\pi^{-1}(1-\ln 2)\} + C \quad \text{as} \quad X \to -\infty \quad \text{for} \quad 0 < Y < 1, \tag{4.5b}$$

where m and C are constants, m being a mass flux. The limit (4.5*b*) takes the solution into the internal outer region, where it must be matched with the shallow-water solution (3.9). Thus matching potentials and mass flux at $x \approx a$ gives

$$\left.\begin{aligned} 1 &= -\frac{2m}{\pi}(1-\ln 2) + C, \\ -p\tan pa &= m/h_1. \end{aligned}\right\} \tag{4.6}$$

To complete the solution, the logarithmic behaviour of the inner solution as $R \to \infty$ needs to be matched to an appropriate solution in an outer external region. This outer region is the quarter-plane $x > a$, $y > 0$, where the potential satisfies conditions (2.6)–(2.8) together with

$$\frac{\partial \phi}{\partial x} = 0 \quad (x = a, \ y > 0) \tag{4.7}$$

$$\phi, \nabla\phi \to 0 \quad \text{as} \quad y \to \infty, \tag{4.8}$$

$$\phi \sim \frac{2m}{\pi}\log r \quad \text{as} \quad r = \{(x-a)^2 + y^2\}^{\frac{1}{2}} \to 0, \tag{4.9}$$

the latter condition ensuring that the solution matches with the inner solution as $R \to \infty$.

The required solution is given in Wehausen & Laitone (1960, p. 547) in the form

$$\phi(x,y) = -\frac{2m}{\pi}\left\{K_0(lr) - 2K\,\mathrm{e}^{-Ky}\int_y^{\infty} \mathrm{e}^{-Kt}K_0(lr')\,\mathrm{d}t + \pi\cot\alpha\,\mathrm{e}^{-l\sin\alpha(x-a)}\,\mathrm{e}^{-Ky}\right\}, \quad (4.10)$$

where

$$\left.\begin{aligned} r' &= \{(x-a)^2+t^2\}^{\frac{1}{2}}, \\ r &= \{(x-a)^2+y^2\}^{\frac{1}{2}}, \\ K &= l\cos\alpha, \end{aligned}\right\} \quad (4.11)$$

and K_0 is the modified Bessel function. It follows that

$$\phi(x,y) = \frac{2m}{\pi}\{\ln Kr + (\alpha-\pi)\cot\alpha + \gamma - \ln(2\cos\alpha)\} + o(1) \quad \text{as} \quad r \to 0, \quad (4.12)$$

where $\gamma = 0.5772...$ is Euler's constant.

A comparison of (4.5*a*) and (4.12) shows that both the logarithmic terms and the constant terms match, provided that

$$C = \frac{2m}{\pi}\left\{\ln\frac{Kh_1}{\pi\cos\alpha} + (\alpha-\pi)\cot\alpha + \gamma\right\}. \quad (4.13)$$

It follows from (4.6) and (4.13) that

$$ph_1\tan pa = \frac{\pi}{2}\left\{1-\gamma-\ln\frac{2Kh_1}{\pi} + (\pi-\alpha)\cot\alpha + \ln(\cos\alpha)\right\}^{-1}. \quad (4.14)$$

Also, since under this approximation $k_2 = K$ and $k_1^2 h_1 = K$, we have

$$(ph_1)^2 = Kh_1 - (lh_1)^2 = Kh_1 - (Kh_1\sec\alpha)^2, \quad (4.15)$$

whilst, since $k_2 < l < k_1$, then, using (4.11),

$$Kh_1 < 1, \quad lh_1 < \cos\alpha. \quad (4.16)$$

Perhaps the simplest method of determining the number of solutions of (4.14) and (4.15) satisfying (4.16) is to fix a, h_1 and K. Then (4.14) is just

$$ph_1\tan pa = \tfrac{1}{2}\pi f(\alpha), \quad (4.17)$$

where

$$f(\alpha) = \frac{\sin\alpha}{A\sin\alpha + (\pi-\alpha)\cos\alpha + \sin\alpha\ln(\cos\alpha)} \quad (4.18)$$

and

$$A = 1-\gamma-\ln\frac{2Kh_1}{\pi} > 0,$$

since

$$\gamma < 1 \quad \text{and} \quad Kh_1 < 1.$$

As α increases from zero to $\frac{1}{2}\pi$, $f(\alpha)$ increases from zero to $+\infty$ at the only value of α in the range for which the denominator of (4.18) vanishes, and then from $-\infty$ to zero as α continues up to $\frac{1}{2}\pi$.

A sketch of the roots of (4.17) is sufficient to deduce that ph_1 *increases* monotonically from $(n-1)\pi h_1/a$ to $n\pi h_1/a$ ($n = 1, 2, \ldots$) as α increases from zero to $\frac{1}{2}\pi$. But (4.15) shows that ph_1 *decreases* monotonically from $(Kh_1-(Kh_1)^2)^{\frac{1}{2}}$ to zero as α increases from zero to $\cos^{-1}Kh_1 < \frac{1}{2}\pi$. Each crossing point gives an edge-wave solution, and it follows

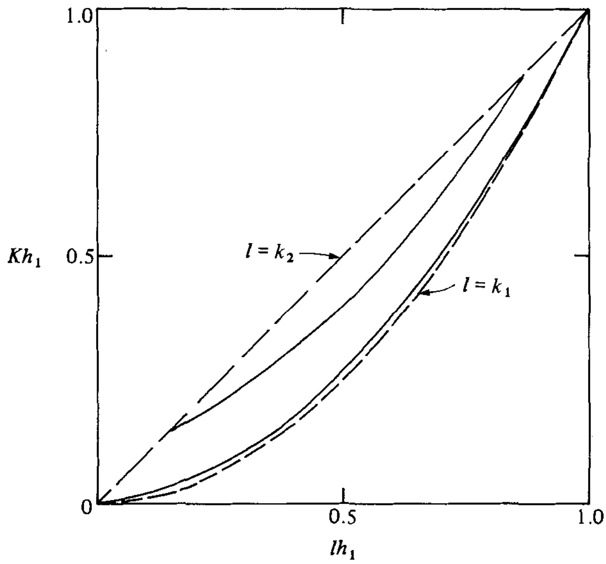

FIGURE 6. The shallow/deep dispersion relation for $a/h_1 = 10$.

that there are precisely n edge-wave solutions if

$$(n-1)\frac{\pi h_1}{a} < (Kh_1 - (Kh_1)^2)^{\frac{1}{2}} < \frac{n\pi h_1}{a}. \tag{4.19}$$

This result is precisely the same as (3.9) for the shallow-water approximation, but now $Kh_1 < 1$ and the variation of $(Kh_1 - (Kh_1)^2)^{\frac{1}{2}}$ with Kh_1, corresponding to (3.10) is given by the semicircle in figure 4. Just as for the shallow-water approximation the permissible values for pa satisfy (3.7).

The precise number of edge-wave solutions for a given value of Kh_1 can be determined from figure 4 and (4.19). In particular it can be seen that the maximum number of modes for a given geometry, occurring when $Kh_1 = \frac{1}{2}$, is given by the greatest integer contained in $1 + a/2\pi h$, whilst for $Kh_1 \to 0, 1$ there is just a single mode.

The variation of lh_1 with Kh_1 can be obtained from computations based on (4.15) and (4.17). Curves showing this variation for a typical value of a/h_1 are presented in figure 6.

Once again results for $Kh_1 \sim 1$ must be treated with caution, since the shallow-water approximation requires $Kh_1 \ll 1$. However, for $Kh_1 \ll 1$ (4.19) shows that it is still possible to have a large number of modes provided that $a/h_1 \gg 1$, a condition which is consistent with shallow-water theory.

There is little difficulty in repeating the analysis of this section under the assumption $Kh_2 = O(1)$ instead of $Kh_2 \gg 1$. All that is required is to modify the solution in the outer region so as to satisfy $\partial\phi/\partial y = 0$ on $y = h_2$ instead of (4.8). This is readily available in, for example, Wehausen & Laitone (1960), but the isolation of the logarithmic singularity from the solution results in a less convenient form for the dispersion relation corresponding to (4.14), from which the number of edge-wave solutions is not readily deduced. Computations, however, confirm that in this case also (3.9) is satisfied and the number of edge-wave modes is finite for all frequencies.

The curve corresponding to the semicircle in figure 4 is determined from the relations

$$(k_1^2-k_2^2)^{\frac{1}{2}} h = (Kh_1 - k_2^2 h_1^2)^{\frac{1}{2}}$$

and

$$K = k_2 \tanh k_2 h_2.$$

Since $K < k_2$ it follows that

$$(k_1^2-k_2^2)^{\frac{1}{2}} h_1 < (Kh_1-(Kh_1)^2)^{\frac{1}{2}},$$

showing that this curve would always lie inside the semicircle describing the shallow/deep case.

5. Full linear theory

The problem using the full linear theory in both water depths h_1 and h_2 can be approached using appropriate eigenfunction expansions as used by Miles (1967) and Grimshaw (1974).

Let k_i $(i = 1, 2)$ be the only positive roots of the equations

$$K = k_i \tanh k_i h_i \quad (i = 1, 2) \tag{5.1}$$

and let α_{in} $(i = 1, 2,\ n = 1, 2, \ldots)$ be the infinite sequence of positive roots taken in ascending order of magnitude of the equations

$$K + \alpha_{in} \tan \alpha_{in} h_i = 0 \quad (i = 1, 2). \tag{5.2}$$

Define

$$\psi_{in}(y) = N_{in}^{-1} \cos \alpha_{in}(h_i - y) \quad (i = 1, 2,\ n = 1, 2, \ldots), \tag{5.3}$$

where

$$N_{in}^2 = \tfrac{1}{2}\{h_i - K^{-1} \sin^2 \alpha_{in} h_i\}, \tag{5.4}$$

and for $n = 0$

$$\psi_{i0}(y) \equiv \chi_i(y) = N_{i0}^{-1} \cosh k_i(h_i - y), \tag{5.5}$$

where $\alpha_{i0} = ik_i$ and

$$N_{i0}^2 = \tfrac{1}{2}\{h_i + K^{-1} \sinh^2 k_i h_i\}. \tag{5.6}$$

Then the set $\{\psi_{in}(y)\}$ $(n = 0, 1, 2, \ldots)$ is orthonormal over $(0, h_i)$ with boundary conditions (2.7) on $y = 0$ and (2.3) on $y = h_i$ (see e.g. Wehausen & Laitone 1960, §16). Separate eigenfunctions can now be constructed appropriate to the regions $0 < x < a$, $0 < y < h_1$, and $x > a$, $0 < y < h_2$. Thus

$$\phi \equiv \phi_1(x, y) = \sum_{n=0}^{\infty} U_{1n} \frac{\cosh \beta_{1n} x}{\beta_{1n} \sinh \beta_{1n} a} \psi_{1n}(y) \tag{5.7}$$

for $0 < x < a$, $0 < y < h_1$, whilst

$$\phi \equiv \phi_2(x, y) = -\sum_{n=0}^{\infty} U_{2n}\, \mathrm{e}^{-\beta_{2n}(x-a)} \beta_{2n}^{-1} \psi_{2n}(y) \tag{5.8}$$

for $x > a$, $0 < y < h_2$. Here

$$\left.\begin{aligned} \beta_{in} &= (\alpha_{in}^2 + l^2)^{\frac{1}{2}} \quad (i = 1, 2,\ n = 1, 2, \ldots), \\ \beta_{10} &= \mathrm{i}p = \mathrm{i}(k_1^2 - l^2)^{\frac{1}{2}}, \\ \beta_{20} &= q = (l^2 - k_2^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.9}$$

The constants U_{in} are just the coefficients in the expansion of the horizontal velocity $U(y)$ across $x = 0, 0 < y < h_1$ in terms of appropriate eigenfunctions, and will be determined by matching both the potentials and $U(y)$ across the interface.

Note that (5.7) satisfies (2.3*a*), (2.4*a*), (2.6) and (2.7), whilst (5.8) satisfies (2.3*b*), (2.6), (2.7) and (2.8). Condition (2.4*b*) will be satisfied in the course of the matching procedure.

There is no reason at this stage for choosing $l < k_1$, but it will turn out that just as for the shallow-water approximation edge-wave solutions require p to be real.

We define $U(y) = \partial\phi/\partial x$ $(x = a,\ 0 < y < h_2)$. Then continuity of $\partial\phi/\partial x$ requires that

$$U(y) = \sum_{n=0}^{\infty} U_{2n}\psi_{2n}(y) = \begin{Bmatrix} \sum_{n=0}^{\infty} U_{1n}\psi_{1n}(y) & (0 < y < h_1), \\ 0 & (h_1 < y < h_2), \end{Bmatrix} \tag{5.10}$$

whilst continuity of ϕ requires that

$$\sum_{n=0}^{\infty} U_{1n}\beta_{1n}^{-1}\coth(\beta_{1n}a)\psi_{1n}(y) = -\sum_{n=0}^{\infty} U_{2n}\beta_{2n}^{-1}\psi_{2n}(y) \quad (0 < y < h_1). \tag{5.11}$$

Since the set $\{\psi_{1n}(y)\}$ is orthonormal over $(0, h_1)$

$$\psi_{2n}(y) = \sum_{m=0}^{\infty} c_{mn}\psi_{1m}(y) \quad (0 < y < h_1),$$

where

$$c_{mn} = \int_0^{h_1} \psi_{1m}(y)\psi_{2n}(y)\,\mathrm{d}y = N_{1m}^{-1}N_{1n}^{-1}\frac{\alpha_{2n}\sin\alpha_{2n}(h_2 - h_1)}{\alpha_{1m}^2 - \alpha_{2n}^2} \quad (m, n = 0, 1, 2, \ldots). \tag{5.12}$$

Multiplication of (5.10) by the orthonormal set $\{\psi_{2m}(y)\}$ and integration over $(0, h_2)$ gives

$$U_{2m} = \sum_{n=0}^{\infty} U_{1n}c_{nm} \quad \left(= \int_0^{h_1} U(y)\psi_{2m}(y)\,\mathrm{d}y\right) \quad (m = 0, 1, 2, \ldots), \tag{5.13}$$

whilst multiplication of (5.11) by $\{\psi_{1m}(y)\}$ and integration over $(0, h_1)$ gives

$$U_{1m}\beta_{1m}^{-1}\coth\beta_{1m}a = -\sum_{n=0}^{\infty} U_{2n}\beta_{2n}^{-1}c_{mn} \quad (m = 0, 1, 2, \ldots). \tag{5.14}$$

Substitution of U_{2m} from (5.13) into (5.14) now gives

$$U_{1n} + \sum_{m=0}^{\infty} A_{nm}U_{1m} = 0 \quad (n = 0, 1, 2, \ldots), \tag{5.15}$$

where

$$A_{nm} = \beta_{1n}\tanh\beta_{1n}a\sum_{r=0}^{\infty}\frac{c_{mr}c_{nr}}{\beta_{2r}}. \tag{5.16}$$

Alternatively, substitution of U_{1m} from (5.14) into (5.13) gives

$$U_{2n} + \sum_{m=0}^{\infty} B_{nm}U_{2m} = 0 \quad (n = 0, 1, 2, \ldots), \tag{5.17}$$

where

$$B_{nm} = \frac{1}{\beta_{2m}}\sum_{r=0}^{\infty}\beta_{1r}\tanh(\beta_{1r}a)c_{rm}c_{rn}. \tag{5.18}$$

The question of existence of edge waves is now one of finding solutions to either of the infinite systems (5.15) or (5.17).

An alternative approach used by Grimshaw (1974) is to derive an integral equation for $U(y)$ by substituting

$$U_{in} = \int_0^{h_i} U(y)\,\psi_{in}(y)\,\mathrm{d}y \quad (i = 1, 2,\ n = 0, 1, 2, \ldots)$$

in (5.11). Since $U(y) = 0$ for $h_1 < y < h_2$ we obtain

$$\int_0^{h_1} K(y, y')\,U(y')\,\mathrm{d}y' = 0 \quad (0 < y < h_1), \tag{5.19}$$

where

$$K(y, y') = \sum_{n=0}^{\infty} \beta_{1n}^{-1}\coth(\beta_{1n}a)\psi_{1n}(y)\,\psi_{1n}(y') + \sum_{n=0}^{\infty} \beta_{2n}^{-1}\,\psi_{2n}(y)\,\psi_{2n}(y'). \tag{5.20}$$

It is now clear why $p = (k_1^2 - l^2)^{\frac{1}{2}}$ must be real. Multiply (5.19) by $U(y)$ and integrate over $(0, h_1)$. The resulting double integral can be expressed as the sum of squares of single integrals if $p = \mathrm{i}p$, corresponding to $l > k_1$, and the only solution is $U(y) = 0$. Hence a necessary condition for edge-wave solutions on the full linear theory is

$$k_2 < l < k_1. \tag{5.21}$$

This argument is due to Grimshaw (1974).

A further approximate solution is provided by extending a procedure first used by Miles (1967) and repeated by Grimshaw (1974) in the present context. The kernel of the integral equation (5.19) is replaced by a new kernel $K_s(y, y')$ obtained by taking the sum in (5.20) from $n = 1$, the two terms corresponding to $n = 0$ now appearing on the right-hand side of (5.19).

Then

$$K_s f_i \equiv \int_0^{h_1} K_s(y, y') f_i(y')\,\mathrm{d}y' = \chi_i(y) \quad (\equiv \chi_i), \tag{5.22}$$

where

$$U(y) = U_{10}\,p^{-1}\cot pa\, f_1(y) - U_{20}\,q^{-1} f_2(y), \tag{5.23}$$

and, multiplying (5.23) by $\chi_i(y)$ and integrating over $(0, h_1)$,

$$U_{i0} = U_{10}\,p^{-1}\cot pa\,\langle f_1, \chi_i\rangle - U_{20}\,q^{-1}\langle f_2, \chi_i\rangle \quad (i = 1, 2),$$

where

$$\langle f, g\rangle = \int_0^{h_1} f(y)\,g(y)\,\mathrm{d}y.$$

These last two equations are consistent provided that

$$(q + S_{22})(p\tan pa - S_{11}) + S_{12}S_{21} = 0, \tag{5.24}$$

where

$$S_{mn} = \langle f_m, \chi_n\rangle \quad (m, n = 1, 2). \tag{5.25}$$

Equation (5.24) must be satisfied for edge-wave solutions. Grimshaw shows that there is at least one solution if l is small enough. Here a different approach is used. From (5.22), (5.25)

$$S_{mn} = \frac{\langle u_m, \chi_n\rangle\,\langle u_n, \chi_m\rangle}{\langle u_m, K_s\,u_n\rangle}, \tag{5.26}$$

and in this form it can be shown that S_{mn} is (i) invariant to a scale transformation of u_m, and (ii) stationary with respect to small variations of $u_m(y)$ about its true value (see Miles 1967, equation (5.2)).

A simple trial function is $u_m = \chi_1$ $(m = 1, 2)$, which gives

$$S_{12} = S_{21} = S_{11}\,c_{00}, \quad S_{22} = S_{11}\,c_{00}^2 \quad \text{and} \quad S_{11}^{-1} = \sum_{r=1}^{\infty} \frac{c_{0r}^2}{\beta_{2r}}.$$

Thus from (5.16) the condition (5.24) reduces simply to

$$1 + A_{00} = 0. \tag{5.27}$$

As with the other approximate solutions, the dispersion relation resulting from (5.12), (5.16) and (5.27) gives n solutions in the range given by the inequality (3.13). Unlike the previous approximations, (5.27) is valid at all frequencies and proves to be a most useful tool in obtaining an estimate of the wavenumber of an edge-wave mode before verification using the full linear theory.

6. Numerical procedure

Consider the infinite system of homogeneous linear equations (5.15). For a finite system of order N of this form, non-trivial solutions exist if and only if the determinant $\Delta_N = |\delta_{nm} + A_{nm}|$ vanishes. It can be proved that an infinite system will behave in the same way, with a bounded solution where $\Sigma_{n=0}^{\infty} |U_{1n}| < \infty$, provided that $\Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} |A_{nm}|$ converges. This result was used by Ursell (1951) when considering trapping modes near a submerged circular cylinder. However, in the present work there is a singularity in the velocity field at the sharp corner on the edge of the shelf. Potential theory shows that near the corner $U(y) \sim (h_1 - y)^{-\frac{1}{3}}$. The eigenfunction expansion (5.8) can describe this singularity provided that $U_{1n} = O(n^{-\frac{2}{3}})$. Thus there can be no non-trivial solution with $\Sigma_{n=0}^{\infty} |U_{1n}| < \infty$, and the abovementioned result is not applicable. Also, it is not possible to infer from the corresponding L_2 theory that *all* solutions of the homogeneous infinite system (5.15) satisfy $\Sigma_{n=0}^{\infty} |U_{1n}|^2 < \infty$.

In the absence of an appropriate theory, extensive numerical checks were carried out to establish the validity of truncating the infinite system. Firstly, the convergence of the sequence $\{\Delta_N\}$ was investigated for arbitrary points in the parameter space, not necessarily corresponding to the zeros of Δ_N. With the equations in the form (5.15) the sequence of determinants did not appear to converge as $N \to \infty$. However, when each row of the matrix was scaled by the factor $(1 + A_{nn})^{-1}$, the coefficient of U_{1n} in (5.15), which had the effect of dividing Δ_N by $\Pi_{n=0}^{N}(1 + A_{nn})$, the modified sequence of determinants did converge. Clearly this procedure affects neither the location of the zeros of Δ_N nor the solution of the truncated system.

The convergence of the sequence of values of lh_1 corresponding to the zeros of Δ_N was also checked. These were located by systematically searching the parameter space using known approximate solutions as a guide. The procedure adopted was to fix the values of μ^2, a/h_1, and Kh_1 and to evaluate Δ_N for increasing values of lh_1 until a zero was found. Once a zero of Δ_N had been located, and checked by taking larger values of N, the solution of the system (5.15) can be determined. It was found that the expected behaviour $|U_{1n}| \propto n^{-\frac{2}{3}}$ was accurately reproduced, and that successive truncations gave a surface profile that changed very little indeed. A value of N as low as 5 gave accurate results in most cases.

In addition to the truncation procedure for the infinite system, careful consideration must also be given to the summation of the series in (5.16). From (5.12) it can be

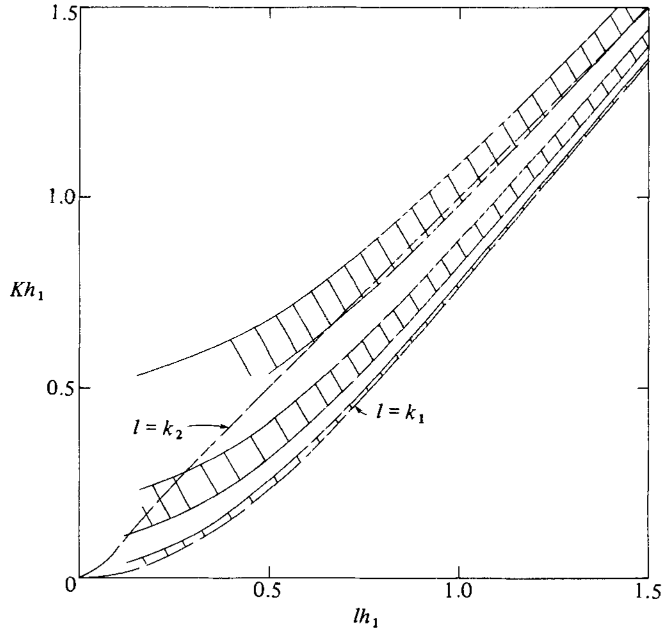

FIGURE 7. Bounds on the longshore wavenumber (indicated by the shaded region) as given by the inequality (6.4) for $\mu^2 = 10$, $a/h_1 = 10$.

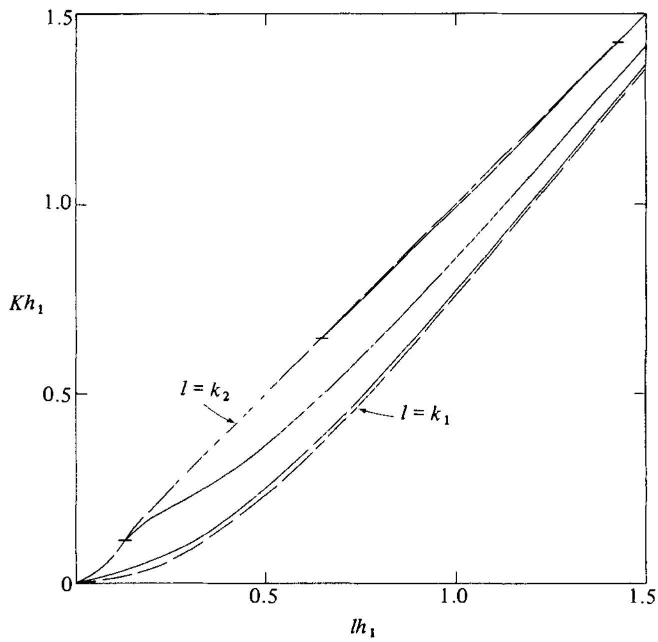

FIGURE 8. The full linear dispersion relation for $\mu^2 = 10$, $a/h_1 = 10$.

seen that each coefficient c_{mn} has a factor $\alpha_{1m}^2 - \alpha_{2n}^2$ in the denominator. Now for large n, $\alpha_{in} \sim n\pi/h_i$, and so the largest terms in the sum will occur when $r = \mu^2 n$, $\mu^2 m$. The summation must therefore be carried out well beyond the term corresponding to $r' = \max\{\mu^2 m, \mu^2 n\}$. In practice this meant summing to about $4r'$ terms to obtain satisfactory convergence.

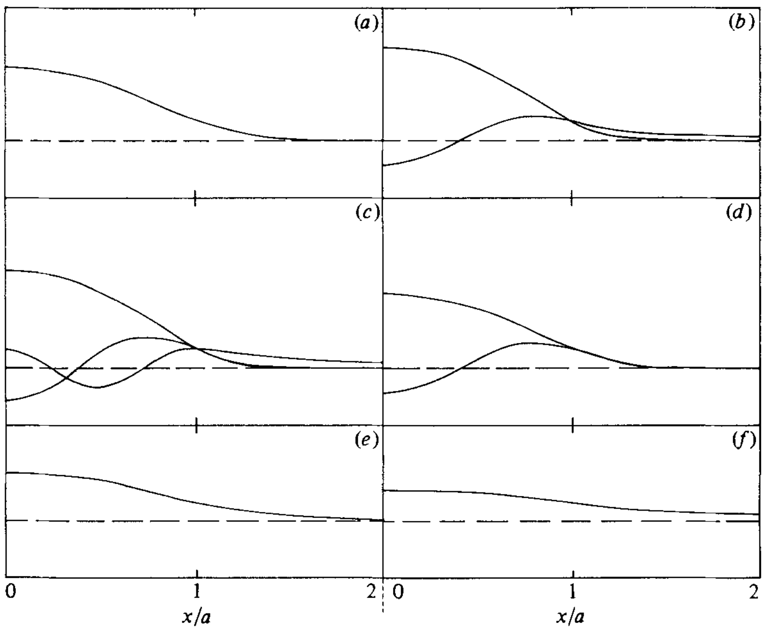

FIGURE 9. Surface profiles for $\mu^2 = 10$, $a/h_1 = 10$ at frequencies Kh_1 of (*a*) 0.05, (*b*) 0.20, (*c*) 1.0, (*d*) 2.0, (*e*) 3.0, (*f*) 4.0.

The search of the parameter space for zeros of the determinant was helped considerably by the work of Jones (1953), who proved the existence of edge-wave solutions in a wide class of problems, including the present geometry. A direct application of his theorems 1′ and 3′ shows that there are precisely n edge-wave solutions when

$$(n-\tfrac{1}{2})\frac{\pi}{a} < (k_1^2-k_2^2)^{\frac{1}{2}} < n\frac{\pi}{a}, \tag{6.1}$$

and *either* $n-1$ *or* n solutions when

$$(n-1)\frac{\pi}{a} < (k_1^2-k_2^2)^{\frac{1}{2}} < (n-\tfrac{1}{2})\frac{\pi}{a}. \tag{6.2}$$

In fact the numerical results confirm that, as for the approximate solutions presented in §§3 and 4, there are exactly n edge-wave solutions when

$$(n-1)\frac{\pi}{a} < (k_1^2-k_2^2)^{\frac{1}{2}} < n\frac{\pi}{a}. \tag{6.3}$$

Using the exact equations (5.1), the expression $(k_1^2-k_2^2)^{\frac{1}{2}}h_1$ is plotted against Kh_1 in figure 4 for two values of μ^2. The curves possess maxima, showing that there are only a finite number of possible modes. In particular, there can be only a single mode at high and low frequencies. For a given shelf width a/h_1, the exact number of modes that exist for a given value of Kh_1 can be read off using the inequalities (6.3).

In addition to the inequalities (6.1) and (6.2), Jones shows that for the nth mode the longshore wavenumber l must lie in the range

$$k_1^2-(n-\tfrac{1}{2})^2\frac{\pi^2}{a^2} < l^2 < k_1^2-(n-1)^2\frac{\pi^2}{a^2} \tag{6.4}$$

or

$$(n-1)\frac{\pi}{a} < p < (n-\tfrac{1}{2})\frac{\pi}{a},$$

in agreement with (3.15) derived using the approximate theories. These bounds on l are drawn in figure 7 for a particular geometry, a case where there are a maximum of three modes. The dashed lines indicate the overall bounds imposed by the requirements that p and q are both real. The actual dispersion relation for this case, determined by locating the zeros of the determinant of $\delta_{mn}+A_{mn}$, is drawn in figure 8. Comparison of this figure with figure 5 shows that, not only does the full linear theory predict only a finite number of edge wave modes at all frequencies, but also the shape of the dispersion curves differs appreciably from that given by shallow-water theory.

Surface profiles are given in figure 9 for a particular geometry at each of six frequencies. All the possible modes at each frequency are shown. Each profile has been normalized to take the same value at the shelf edge ($x = a$). It is interesting to note that the fundamental mode (which exists at all frequencies) has the largest amplitude throughout, suggesting that this will be the dominant mode if such edge waves were to be excited through nonlinear interactions by waves incident from the deep water. This is the behaviour found by Guza & Davis (1974) when considering edge-wave excitation on a sloping beach.

7. Conclusion

The existence of edge-wave modes on a horizontal shelf adjoining a straight coastline has been investigated. Using a full linear theory it has been shown that only a finite number of modes are possible at all frequencies. For sufficiently low or high frequencies there exists only one mode, with a single maximum in the number of modes at an intermediate frequency. This behaviour is in contrast with previously known shallow-water theory, which predicts that the number of modes will increase indefinitely with increasing frequency.

Attention has been drawn to the work of Jones (1953), which, for a fixed geometry, predicts the number of modes that may be excited at any given frequency. For certain ranges of the frequency, Jones' theory only predicts to within one the number of possible modes, whilst over the remaining ranges the exact number is predicted (see the inequalities (6.1) and (6.2)). However, the numerical evidence presented here suggests that the exact number of modes may be predicted at all frequencies using the inequality (6.3). This inequality is in agreement with shallow-water theory and with a new approximate theory which assumes shallow water above the shelf but infinitely deep water elsewhere. These theories enable the character of the solution at the extremes of the outer depth to be assessed. The dispersion relation for the full linear theory can be calculated by locating the zeros of an infinite determinant. This task is made relatively straightforward, as the number of modes at a given frequency is known, whilst bounds for the longshore wavenumber l are given by the inequality (6.4). The two approximate theories provide useful checks on the computations in certain limiting cases, while the approximation leading to (5.27) turns out to be particularly useful.

Long waves trapped on a continental shelf, which can be approximated by the rectangular geometry considered in the present work, have been observed by Snodgrass *et al.* (1962). It was these observations that motivated their original derivation of the shallow-water theory described in the present §3. For such oceanographic scales shallow-water theory provides an adequate description of the motion. There have been many observations of short edge waves on a sloping beach (see e.g. Huntley & Bowen 1973). However, the authors are not aware of any observations of short waves involving a bottom geometry that is approximately rectangular and where the present full linear theory would be applicable.

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REFERENCES

ECKART, C. 1951 Surface waves in water of variable depth. *Marine Phys. Lab. Scripps Inst. Oceanogr. Wave Rep.* 100-99.

GRIMSHAW, R. 1974 Edge waves: a long wave theory for oceans of finite depth. *J. Fluid Mech.* **62**, 775–791.

GUZA, R. T. & DAVIS, R. E. 1974 Excitation of edge waves by waves incident on a beach. *J. Geophys. Res.* **79**, 1285–1291.

HUNTLEY, D. A. -& BOWEN, A. J. 1973 Field observations of edge waves. *Nature* **243**, 160–162.

JONES, D. S. 1953 The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains. *Proc. Camb. Phil. Soc.* **49**, 668–684.

LONGUET-HIGGINS, M. S. 1967 On the trapping of wave energy round islands. *J. Fluid Mech.* **29**, 781–821.

MILES, J. W. 1967 Surface-wave scattering for a shelf. *J. Fluid Mech.* **28**, 755–767.

NEWMAN, J. N., SORTLAND, B. & VINJE, T. 1983 The added mass and damping of rectangular bodies close to the free surface. Private communication.

ROSEAU, M. 1958 Short waves parallel to the shore over a sloping beach. *Commun. Pure Appl. Maths* **11**, 433–493.

SNODGRASS, F. E., MUNK, W. H. & MILLER, G. R. 1962 Long period waves over California's Continental Borderland. Part I: Background Spectra. *J. Mar. Res.* **20**, 3–30.

STOKES, G. G. 1846 Report on recent researches in hydrodynamics. *Brit. Assn Rep.*

SUMMERFIELD, W. 1972 Circular islands as resonators of long-wave energy. *Phil. Trans. R. Soc. Lond.* A **272**, 361–402.

TUCK, E. O. 1976 Some classical water-wave problems in varying depth. In *Proc. IUTAM Symp. Waves on Water of Varying Depth, Canberra.*

TUCK, E. O. 1980 The effect of a submerged barrier on the natural frequencies and radiation damping of a shallow basin connected to open water. *J. Austral. Math. Soc.* (*Series B*) **22** 104–128.

URSELL, F. 1951 Trapping modes in the theory of surface waves. *Proc. Camb. Phil. Soc.* **47**, 347–358.

URSELL, F. 1952 Edge waves on a sloping beach. *Proc. R. Soc. Lond.* A **214**, 79–97.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. In *Handbuch der Physik* (ed. W. Flügge), vol. 9, pp. 446–778. Springer.